Lecture 1.

Problem 13

Suppose A and u_0 are given numbers and we want to solve the differential equation

$$\frac{du}{\partial t} = Au \quad , \quad t > 0$$

with the initial condition

$$u(0) = u_0 .$$

We know the answer is $u(t) = u_0 e^{tA}$.

We want to put the problem in a more general framework. Let X be a Banach space. We can define functions of a real variable t with values in X. Such a function $u(t)$ is continuous at t_0 if for any $\varepsilon > 0$ given there is a corresponding $\delta > 0$ so that $|t - t_0| < \delta$ $\Rightarrow$ $\| u(t) - u(t_0) \| < \varepsilon$. Likewise $u(t)$ is differential at t_0 if there is an element $v \in X$

so that given $\varepsilon > 0$, there is a $\delta > 0$ so that

$$|t - t_0| < \delta \Rightarrow \left\| \frac{u(t) - u(t_0)}{t - t_0} - v \right\| < \varepsilon.$$

In such a case we denote v by $u'(t_0)$ or $\frac{du}{dt}(t_0)$. Of course, when u is differentiable at t_0, we have $\|u(t) - u(t_0)\|$

$$= \left\| \frac{u(t) - u(t_0)}{(t - t_0)} (t - t_0) \right\| \leq (\|v\| + \varepsilon)|t - t_0|$$

whenever $|t - t_0| < \delta$. So u differentiable at t_0 $\Rightarrow$ u is continuous at t_0.

Now let A be any operator in $B(X)$, and let $u_0 \in X$. Then the equations

$$u'(t) = Au \qquad , t > 0$$

$$u(0) = u_0$$

make sense. So we have an initial value problem in a Banach space X. Does it have a solution?

Our guess should be that it is an appropriate analogue to $u_0 e^{tA}$ (which of course is the same as $e^{tA} u_0$ for real numbers).

What should e^{tA} mean?

For $A \in \mathbb{R}$,

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k,$$

which converges for all A.

But now if $A \in B(X)$, $\sum_{k=0}^{n} \frac{1}{k!} t^k A^k \in B(X)$

for all n. Moreover, if $n > m$,

$$\left\| \sum_{k=0}^{n} \frac{1}{k!} t^k A^k - \sum_{k=0}^{m} \frac{1}{k!} t^k A^k \right\| = \left\| \sum_{k=m+1}^{n} \frac{1}{k!} t^k A^k \right\|$$

$$\leq \sum_{k=m+1}^{n} \frac{1}{k!} |t|^k \|A\|^k \to 0 \text{ as } m \text{ (and hence } n)$$

$\to \infty$. So $\sum_{k=0}^{n} \frac{1}{k!} t^k A^k$ is a Cauchy sequence in $B(X)$ and so it makes sense to define

$$e^{tA} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k.$$

Moreover, $\left\| \sum_{k=0}^{n} \frac{1}{k!} t^k A^k \right\| \leq \sum_{k=0}^{n} \frac{1}{k!} |t|^k \|A\|^k$

$$\leq \sum_{k=0}^{\infty} \frac{1}{k!} |t|^k \|A\|^k$$

$$= e^{|t| \|A\|} \quad \forall n.$$

$$\therefore \|e^{tA}\| \leq e^{|t|\|A\|}.$$

Now we have $e^{tA} u_0$ well defined as a map from $[0, \infty) \to X$. Does $e^{tA} u_0$ give a solution to the equation at hand?

Set $u(t) = e^{tA} u_0$

$$\frac{u(t+h) - u(t)}{h} = \frac{e^{(t+h)A} u_0 - e^{tA} u_0}{h}$$

$$= \frac{e^{tA + hA} u_0 - e^{tA} u_0}{h}$$

$$\stackrel{?}{=} \frac{e^{tA} e^{hA} u_0 - e^{tA} u_0}{h} = e^{tA} \left(\frac{e^{hA} u_0 - u_0}{h} \right)$$

$$= \cdots$$

$$e^{tA}\left[\frac{e^{hA}-I}{h}\right]u_0$$

provided $e^{tA+hA} = e^{tA}\cdot e^{hA}$ (note: since $e^{tA+hA} = e^{hA+tA}$, we would automatically have $e^{hA}\cdot e^{tA} = e^{tA}\cdot e^{hA}$)

We'll see about this shortly.

So assuming $e^{tA+hA} = e^{tA}\cdot e^{hA}$, we have

$$\frac{u(t+h)-u(t)}{h} = \left[\frac{e^{hA}-I}{h}\right]e^{tA}u_0$$

$$= \left[\frac{e^{hA}-I}{h}\right]u(t)$$

So $\frac{u(t+h)-u(t)}{h} - Au(t)$

$$= \left[\frac{e^{hA}-I}{h} - A\right]u(t)$$

Now $$\frac{e^{hA}-I}{h} = \sum_{k=1}^{\infty}\frac{1}{k!}h^{k-1}A^k$$

$$\therefore \left\|\frac{e^{hA}-I}{h} - A\right\| = \left\|\sum_{k=2}^{\infty}\frac{1}{k!}h^{k-1}A^k\right\|$$

$$\leq \sum_{k=2} \frac{1}{k!} |h| \quad \|A\|$$

$$= \sum_{k=1}^{\infty} \frac{1}{k!} |h|^{k-1} \|A\|^{k} - \|A\|$$

$$= \frac{1}{|h|} \sum_{k=1}^{\infty} \frac{1}{k!} |h|^{k} \|A\|^{k} - \|A\|$$

$$= \frac{1}{|h|} \left(\sum_{k=0}^{\infty} \frac{1}{k!} |h|^{k} \|A\|^{k} - 1 \right) - \|A\|$$

$$= \frac{e^{|h|\|A\|} - 1}{|h|} - \|A\| \to 0 \quad \text{as } |h| \to 0.$$

So we need only show that $e^{B+C} = e^{B} \cdot e^{C}$.

This holds if $BC = CB$ (which clearly tA and hA do)

We need to establish that $e^{B+C} = e^B e^C$ if $BC = CB$.

Note that since $BC = CB$, $(B+C)^k = \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} B^m C^{k-m}$

$$\text{So } \sum_{k=0}^{N} \frac{1}{k!}(B+C)^k = \sum_{k=0}^{N}\sum_{m=0}^{N} \frac{1}{m!(k-m)!} B^m C^{k-m}$$

$$= \sum_{m+n \le N} \frac{1}{m!n!} B^m C^n$$

$$\text{So } \sum_{m=0}^{N} \frac{1}{m!} B^m \cdot \sum_{n=0}^{N} \frac{1}{n!} C^n - \sum_{k=0}^{N} \frac{1}{k!}(B+C)^k$$

$$= \sum_{\substack{m,n \le N \\ m+n > N}} \frac{1}{m!n!} B^m C^n .$$

$$\text{So } \left\| \sum_{m=0}^{N} \frac{1}{m!} B^m \cdot \sum_{n=0}^{N} \frac{1}{n!} C^n - \sum_{k=0}^{N} \frac{1}{k!}(B+C)^k \right\|$$

$$= \left\| \sum_{\substack{m,n \le N \\ m+n > N}} \frac{1}{m!n!} B^m C^n \right\|$$

$$\le \sum_{\substack{m,n \le N \\ m+n > N}} \frac{1}{m!n!} \|B\|^m \|C\|^n = \sum_{m=0}^{N} \frac{1}{m!} \|B\|^m \cdot \sum_{n=0}^{N} \frac{1}{n!} \|C\|^n$$

$$- \sum_{k=0}^{N} \frac{1}{k!} (\|B\| + \|C\|)^k$$

$$\rightarrow e^{\|B\|} \cdot e^{\|C\|} - e^{\|B\| + \|C\|} = 0$$

as $N \rightarrow \infty$.

Lecture 10 : 02/19/07

Problem 14

Is the solution $u(t) = e^{tA} u_0$ of $u' = Au$

from last time unique?

Suppose there were two solutions.

Their difference (call it u) would satisfy

$$u' = Au$$

$$u(0) = 0.$$

So it would be the case that

$$e^{-tA}(u' - Au) = 0$$

(in the real number case this equation is

$$\frac{d}{dt}(e^{-tA} u) = 0.$$

Here we must be careful.

Set $v(t) = e^{-tA} u(t)$

Then $$\frac{v(t+h) - v(t)}{h} = \frac{e^{-(t+h)A} u(t+h) - e^{-tA} u(t)}{h}$$

$$= e^{-(t+h)A}\left[\frac{u(t+h) - u(t)}{h}\right] + \frac{e^{-(t+h)A} u(t) - e^{-tA} u(t)}{h}$$

$$= e^{-(t+h)A}\left[\frac{u(t+h)-u(t)}{h}\right] + \left[\frac{e^{-hA}-I}{h}\right](e^{-tA}u(t))$$

$$= e^{-(t+h)A}\left[\frac{u(t+h)-u(t)}{h}\right] + \left[\frac{e^{-hA}-I}{h}\right]v(t)$$

$$\to e^{-tA}u'(t) + (-A)e^{-tA}u(t)$$

as $h \to 0$.

So we get $\frac{d}{dt}\left(e^{-tA}u(t)\right)$

$$= e^{-tA}(u'(t) - Au(t))$$

$$= 0$$

Now let $f \in X'$.

Set $F(t) = f(e^{-tA}u)$.

$$\frac{F(t+h)-F(t)}{h} = \frac{f\left(e^{-(t+h)A}u(t+h)\right) - f(e^{-tA}u(t))}{h}$$

$$= f\left(\frac{e^{-(t+h)A}u(t+h) - e^{-tA}u(t)}{h}\right)$$

$$\to f\left(e^{-tA}(u'(t) - Au(t))\right) = f(0) = 0$$

for $t > 0$.

So $F(t)$ is differentiable for $t > 0$,

$F'(t) = 0$, F is continuous for $t \geq 0$

and $F(0) = F(e^{-0 \cdot A} u(0)) = F(0) = 0$.

So $F \equiv 0$. $\therefore f(e^{-tA} u(t)) \equiv 0$ for

all $f \in X'$. $\therefore e^{-tA} u(t) = 0$ in X.

$\therefore u(t) = e^{tA} e^{-tA} u(t) = e^{tA}(0) = 0$.

Suppose now that A is not bounded, but

rather a closed linear operator on X with

domain $D(A)$ that is dense in X.

The equation $u'(t) = Au(t)$ continuous

to make sense now so long as we stipulate

that $u(t) \in D(A)$ for $t > 0$.

Theorem 10.1: Let A be a closed linear operator

with dense domain $D(A)$ on X having the interval

$[b,\infty)$ in its resolvent set $\rho(A)$, where $b \geq 0$, and such that there is a constant a satisfying

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{a+\lambda}, \quad \lambda \geq b.$$

Then there is a family $\{E_t\}$ of operators in $B(X)$, $t \geq 0$ so that:

(a) $E_s E_t = E_{s+t}$, $s \geq 0$, $t \geq 0$

(b) $E_0 = I$

(c) $\|E_t\| \leq e^{-at}$, $t \geq 0$

(d) $E_t x$ is continuous in $t \geq 0$ for each $x \in X$

(e) $E_t x$ is differentiable in $t \geq 0$ for each $x \in D(A)$ and $\dfrac{dE_t x}{dt} = AE_t x$

(f) $E_t(\lambda - A)^{-1} = (\lambda - A)^{-1}E_t$, $\lambda \geq b$, $t \geq 0$.

Suppose $u_0 \in D(A)$. Set $u(t) = E_t u_0$.

Claim: $u(t) \in D(A)$. $(E_t u_0 = E_t(b-A)^{-1}(b-A)u_0$

$= (b-A)^{-1} E_t (b-A) u_0 \in D(A)$ (since $(b-A)^{-1} : X \to D(A)$) Then $\frac{dE_t u_0}{dt} = A E_t u_0 \Rightarrow \frac{d}{dt}(u(t)) = Au(t)$ by (e). (b) $\Rightarrow E_0 u_0 = I u_0 = u_0$.

A one-parameter family $\{E_t\}$ of operators satisfying (a) and (b) is called a semi-group. The operator A is called its infinitesimal generator.

Before proving Theorem 10.1, we establish

<u>Lemma 10.2</u>. Let D be dense in X, and let $\{B_\lambda\}$ be a family of operators in $B(X)$ satisfying $\|B_\lambda\| \le M$, $\lambda \ge K$.

If $B_\lambda x$ converges as $\lambda \to \infty$, for each $x \in D$, then there is a B in $B(X)$ such that

$$\|B\| \le M$$

and $B_\lambda x \to Bx$ as $\lambda \to \infty$.

Proof: Let $\varepsilon > 0$ be given. Let $x \in X$. $\exists\ \tilde{x} \in D$

$$\ni \|x - \tilde{x}\| < \frac{\varepsilon}{3M}.$$

Then $\|B_\lambda x - B_\mu x\| \le \|B_\lambda(x - \tilde{x})\|$

$+ \|B_\lambda \tilde{x} - B_\mu \tilde{x}\| + \|B_\mu(\tilde{x} - x)\|$

$$< \frac{2\varepsilon}{3} + \|B_\lambda \tilde{x} - B_\mu \tilde{x}\|$$

We now take λ, μ large enough so that

$$\|B_\lambda \tilde{x} - B_\mu \tilde{x}\| < \varepsilon/3.$$

So $\lim_{\lambda \to \infty} B_\lambda x$ exists for all $x \in X$.

$\therefore$ Let B be defined by $Bx = \lim_{\lambda \to \infty} B_\lambda x$

B is linear on X.

Let $x \in X$ be fixed.

Let $\varepsilon > 0$ be given. $\exists\ \bar{\lambda} \ni \lambda \ge \bar{\lambda} \Rightarrow \|Bx - B_\lambda x\| < \varepsilon$.

So $\|Bx\| \le \|Bx - B_\lambda x\| + \|B_\lambda x\| < \varepsilon + M\|x\|$.

ε arbitrary $\Rightarrow \|Bx\| \le M\|x\| \Rightarrow B$

is bounded and $\|B\| \le M$.

Proof of Theorem 10.1 Assume first that $a > 0$.

Set $\qquad A_\lambda = \lambda A(\lambda - A)^{-1}, \quad \lambda \geq b.$

Claim: (1) $A_\lambda \in B(X), \quad \lambda \geq b.$

(2) $\|e^{tA_\lambda}\| \leq \exp\left(\frac{-at\lambda}{a+\lambda}\right), \quad t \geq 0, \quad \lambda \geq b$

(3) $A_\lambda x \to Ax$ as $\lambda \to \infty, \quad x \in D(A)$

(4) $e^{tA_\lambda}x \to E_t x$ as $\lambda \to \infty, \ x \in X, \ t \geq 0.$

(The last assertion states for $t \geq 0$ and $x \in X$,

$e^{tA_\lambda}x$ converges to a limit in X as $\lambda \to \infty$.

We define this limit to be $E_t x$. E_t is a linear

operator. (2) and Lemma 10.2 $\Rightarrow$ $E_t \in B(X)$.

Moreover, $\|E_t x\| \leq \|e^{tA_\lambda}x\| + \|(E_t - e^{tA_\lambda})x\|$

$$\leq \exp\left(\frac{-at\lambda}{a+\lambda}\right)\|x\| + \|(E_t - e^{tA_\lambda})x\|$$

Letting $\lambda \to \infty$, by (4) we get $\|E_t x\| \leq \exp(-at)\|x\|$

$\Rightarrow$ (c)

We show the remainder of (a) - (f).

(a): $\| E_{s+t} x - E_s E_t x \|$

$$\leq \| (E_{s+t} - e^{(s+t)A_\lambda}) x \| + $$

$$+ \| (e^{(s+t)A_\lambda} - e^{sA_\lambda} E_t) x \|$$

$$+ \| (e^{sA_\lambda} - E_s) E_t x \|$$

$$= \| (E_{s+t} - e^{(s+t)A_\lambda}) x \|$$

$$+ \| e^{sA_\lambda} (e^{tA_\lambda} - E_t) x \|$$

$$+ \| (e^{sA_\lambda} - E_s) E_t x \|$$

$$\leq \| (E_{s+t} - e^{(s+t)A_\lambda}) x \|$$

$$+ \exp\left(\frac{-as\lambda}{a+\lambda}\right) \| (e^{tA_\lambda} - E_t) x \|$$

$$+ \| (e^{sA_\lambda} - E_s) E_t x \|$$

$\to 0$ as $\lambda \to \infty$.

So $E_{s+t} x = E_s E_t x \quad \forall x$.

$\therefore E_{s+t} = E_s E_t$

(d): $e^{tA_\lambda} x - e^{t_0 A_\lambda} x = \int_{t_0}^{t} (e^{sA_\lambda} x)' ds$ (FTC)

$$= \int_{t_0}^{t} e^{sA_\lambda} A_\lambda \dot{x} \, ds$$

So

$$\| e^{tA_\lambda} x - e^{t_0 A_\lambda} x \|$$

$$\le \int_{t_0}^{t} \| e^{sA_\lambda} A_\lambda x \| ds \le \int_{t_0}^{t} \| A_\lambda x \| ds = (t - t_0) \| A_\lambda x \|$$

Assume now that $x \in D(A)$.

Let $\lambda \to \infty$. $(3) + (4) \Rightarrow$

$$\| E_t x - E_{t_0} x \| \le (t - t_0) \| A x \|$$

$\therefore E_t x \to E_{t_0} x$ as $t \to t_0$, for $x \in D(A)$

(Now let $x \in X$ be arbitrary.

Let $\hat{x} \in D(A) \ni \| x - \hat{x} \| < \delta$..

Then $\| E_t x - E_{t_0} x_0 \| \le \| E_t x - E_t \hat{x} \| + \| E_t \hat{x} - E_{t_0} \hat{x} \|$

$+ \| E_{t_0} \hat{x} - E_{t_0} x \| \le e^{-at} \| x - \hat{x} \| + \| E_t \hat{x} - E_{t_0} \hat{x} \| + e^{-at_0} \| x - \hat{x}_0 \|$

$\le 2 \| x - \tilde{x} \| + \| E_t \hat{x} - E_{t_0} \tilde{x} \|$)

So $E_t x$ is continuous in t for $x \in X$.

We have

$$e^{tA_\lambda} x - e^{t_0 A_\lambda} x = \int_{t_0}^{t} e^{sA_\lambda} A_\lambda x \, ds$$

If $x \in D(A)$, ~~[illegible]~~ $e^{tA_\lambda x} \to E_t x$ and $e^{-t_0 A_\lambda x} \to E_{t_0} x$

as $\lambda \to \infty$.

Now $\left\| \int_{t_0}^{t} e^{sA_\lambda} A_\lambda x \, ds - \int_{t_0}^{t} E_s A x \, ds \right\|$

$$\le \int_{t_0}^{t} \| e^{sA_\lambda} A_\lambda x - E_s A x \| \, ds$$

$$\le \int_{t_0}^{t} \| e^{sA_\lambda} A_\lambda x - e^{sA_\lambda} A x \| \, ds$$

$$+ \int_{t_0}^{t} \| e^{sA_\lambda} A x - E_s A x \| \, ds$$

$$\le \int_{t_0}^{t} \exp\left(\frac{-as\lambda}{a+\lambda}\right) \| A_\lambda x - A x \| \, ds$$

$$+ \int_{t_0}^{t} \| e^{sA_\lambda} A x - E_s A x \| \, ds .$$

The first integral clearly $\to 0$ as $\lambda \to \infty$.

For the second, we have for each $s \in [t_0, t]$

$$\| e^{sA_\lambda} A x - E_s A x \| \to 0 \quad \text{as } \lambda \to \infty$$

But now $\| e^{sA_\lambda} A x - E_s A x \|$

$$\le \left[e^{-\left(\frac{as\lambda}{a+\lambda}\right)} + e^{-as} \right] \| A x \| \le 2 \| A x \|$$

$\forall\, s \in [t_0, t]$. So the Dominated Convergence

Theorem gives that the second integral tends to zero.

So we get

$$E_t x - E_{t_0} x = \int_{t_0}^{t} E_s A x \, ds, \qquad x \in D(A)$$

So now $\dfrac{E_t x - E_{t_0} x}{t - t_0} - E_{t_0} A x = \dfrac{1}{t - t_0} \displaystyle\int_{t_0}^{t} (E_s A x - E_{t_0} A x) \, ds$

So $\left\| \dfrac{E_t x - E_{t_0} x}{t - t_0} - E_{t_0} A x \right\| \leq \dfrac{1}{t - t_0} \displaystyle\int_{t_0}^{t} \| E_s A x - E_{t_0} A x \| \, ds$

Let $\varepsilon > 0$ be given. $\exists \ \delta > 0 \ni$ if $|s - t_0| < \delta$,

$\| E_s A x - E_{t_0} A x \| < \varepsilon$. Let $t < t_0 + \delta$.

Then $\left\| \dfrac{E_t x - E_{t_0} x}{t - t_0} - E_{t_0} A x \right\| < \varepsilon$.

So $\dfrac{d}{dt} E_t x = E_t A x, \qquad x \in D(A)$

So, now assuming (f), we get

$$A E_t x = (b - (b - A)) E_t x$$

$$= (b E_t x) - (b - A) E_t (b - A)^{-1} (b - A) x$$

$$= b E_t x - (b - A)(b - A)^{-1} E_t (b - A) x$$

$$= bE_t x - E_t (b-A)x$$

$$= bE_t x - bE_t \lambda + E_t Ax = E_t Ax$$

So $\frac{d}{dt}\left(E_t x\right) = E_t Ax = AE_t x$

To see (f), note that if $x \in D(A)$ and $\lambda \geq b$

$$A(A-\lambda)^{-1}x = (A-\lambda+\lambda)(A-\lambda)^{-1}x$$

$$= (A-\lambda)(A-\lambda)^{-1}x + \lambda(A-\lambda)^{-1}x$$

$$= x + (A-\lambda)^{-1}\lambda x$$

$$= (A-\lambda)^{-1}[(A-\lambda)x + \lambda x]$$

$$= (A-\lambda)^{-1}Ax$$

Let $\mu \geq b$. So $A_\lambda (A-\mu)^{-1}x = \lambda A(A-\lambda)^{-1}(A-\mu)^{-1}x$

and let $x \in X$.

$$= \lambda A(A-\mu)^{-1}(A-\lambda)^{-1}x$$

$$= (A-\mu)^{-1}\lambda A(A-\lambda)^{-1}x$$

$$= (A-\mu)^{-1}A_\lambda x$$

Likewise $A_\lambda^2 (A-\mu)^{-1}x = \lambda A(A-\lambda)^{-1}\lambda A(A-\lambda)^{-1}(A-\mu)^{-1}x$

$$= \lambda A(A-\lambda)^{-1}\lambda A(A-\mu)^{-1}(A-\lambda)^{-1}x$$

$$= \lambda A (A-\lambda) \quad (A-\mu) \quad \lambda A (A-\lambda)\, x$$

$$= \lambda A (A-\mu)^{-1} (A-\lambda)^{-1} A_\lambda x$$

$$= (A-\mu)^{-1} \lambda A (A-\lambda)^{-1} A_\lambda x = (A-\mu)^{-1} A_\lambda^2 x$$

So we have

$$\sum_{k=1}^{N} \frac{1}{k!} t^k A_\lambda^k (A-\mu)^{-1} x = (A-\mu)^{-1} \sum_{k=1}^{N} \frac{1}{k!} t^k A_\lambda^k x$$

for $\lambda, \mu \geq b$, $x \in X$.

$$\therefore \quad e^{tA_\lambda} (\mu - A)^{-1} x = (\mu - A)^{-1} e^{tA_\lambda} x, \quad \lambda, \mu \geq b, \ x \in X.$$

Let $\lambda \to \infty$.

$$E_t (\mu - A)^{-1} x = (\mu - A)^{-1} E_t x, \quad x \in X,$$

giving (f).

Finally, $e^{0 \cdot A_\lambda} = I$ for all λ. Hence $E_0 = I$.

So we need to establish (1)–(4).

(1): $A_\lambda = \lambda A (\lambda - A)^{-1} \in B(X)$.

$$\lambda(\lambda - A)^{-1} = (\lambda - A + A)(\lambda - A)^{-1}$$

$$= (\lambda - A)(\lambda - A)^{-1} + A(\lambda - A)^{-1}$$

$$= I + A(A - \lambda)^{-1}$$

$$\therefore \; A(A - \lambda)^{-1} = -I + \lambda(\lambda - A)^{-1}$$

$$\Rightarrow \lambda A(A - \lambda)^{-1} = -\lambda + \lambda^2(\lambda - A)^{-1} \in B(X).$$

(2): $\|e^{tA_\lambda}\| \leq \exp\left(\frac{-at\lambda}{a+\lambda}\right)$, $t \geq 0$, $\lambda \geq b$.

$$A_\lambda = -\lambda + \lambda^2(\lambda - A)^{-1}$$

$$tA_\lambda = -t\lambda + t\lambda^2(\lambda - A)^{-1}$$

So $$e^{tA_\lambda} = e^{-t\lambda I} e^{t\lambda^2(\lambda - A)^{-1}}$$

$$= e^{-t\lambda} e^{t\lambda^2(\lambda - A)^{-1}}$$

$$\therefore \|e^{tA_\lambda}\| \leq e^{-t\lambda} e^{t\lambda^2\|(\lambda - A)^{-1}\|}$$

$$= e^{-t\lambda} e^{t\lambda^2/a+\lambda}$$

$$= e^{-t\lambda + \frac{t\lambda^2}{a+\lambda}}$$

$$= e^{\left[\frac{-at\lambda - t\lambda^2 + t\lambda^2}{a+\lambda}\right]} = e^{\frac{-at\lambda}{a+\lambda}} \leq 1.$$

(3) : $A_\lambda x \to Ax$ as $\lambda \to \infty$ for $x \in D(A)$.

Now $A(A-\lambda)^{-1} = -I + \lambda(\lambda - A)^{-1}$

$$\Rightarrow \quad \| A(\lambda - A)^{-1} \| \leq \| -I \| + \lambda \| (\lambda - A)^{-1} \|$$

$$\leq 1 + \frac{\lambda}{a+\lambda} \leq 2.$$

Now for $x \in D(A)$,

$$\| A(\lambda - A)^{-1} x \| = \| (\lambda - A)^{-1} A x \| \leq \frac{\| Ax \|}{a+\lambda} \to 0$$

as $\lambda \to \infty$.

So $A(\lambda - A)^{-1} x \to 0$ for $x \in D(A)$ and

$$\| A(\lambda - A)^{-1} \| \leq 2 \quad \forall \lambda \geq b.$$

$\therefore$ $A(\lambda - A)^{-1} x \to 0 \quad \forall x \in X$ by Lemma 10.2.

So $\lambda(\lambda - A)^{-1} x = x + A(\lambda - A)^{-1} x$

$$\to x + 0 \quad \text{as } \lambda \to \infty$$

for all $x \in X$.

So $\lambda(\lambda - A)^{-1} A x \to Ax$ as $\lambda \to \infty$

for $x \in D(A)$ $\therefore$ $\lambda A(\lambda - A)^{-1} x \to Ax$ for $x \in D(A)$

as $\lambda \to \infty$

(4) : $e^{tA_\lambda} x \to E_t x$ as $\lambda \to \infty$, $x \in X$, $t \geq 0$

Let λ, μ be any two numbers $\geq b$.

$$A_\lambda = -\lambda + \lambda^2(\lambda - A)^{-1}$$

$$A_\mu = -\mu + \mu^2(\mu - A)^{-1}$$

$$\Rightarrow \quad A_\lambda A_\mu = A_\mu A_\lambda$$

Set $V_s = \exp[stA_\lambda + (1-s)tA_\mu]$, $s \in [0,1]$

Let $v(s) = V_s x = \exp[tA_\mu]\exp[st(A_\lambda - A_\mu)]x$

$$= \exp[st(A_\lambda - A_\mu)]\exp[tA_\mu]x$$

Then $v'(s) = t(A_\lambda - A_\mu)\exp[st(A_\lambda - A_\mu)]\exp[tA_\mu]x$

$$= t(A_\lambda - A_\mu)V_s x = t(A_\lambda - A_\mu)v(s)$$

Hence $[\exp(tA_\lambda) - \exp(tA_\mu)]x$

$$= v(1) - v(0)$$

$$= \int_0^1 v'(s)\,ds = t(A_\lambda - A_\mu)\int_0^1 v(s)\,ds$$

$$= t\int_0^1 V_s(A_\lambda - A_\mu)x\,ds$$

Now $stA_\lambda + (1-s)tA_\mu =$

$$= st(-\lambda + \lambda^2(\lambda - A)^{-1}) + (1-s)t(-\mu + \mu^2(\mu - A)^{-1})$$

$$= -st\lambda - (1-s)t\mu + st\lambda^2(\lambda - A)^{-1} + (1-s)t\mu^2(\mu - A)^{-1}$$

$$\Rightarrow V_s = \exp(-st\lambda - (1-s)t\mu)\exp[st\lambda^2(\lambda - A)^{-1} + (1-s)t\mu^2(\mu - A)^{-1}]$$

$$\Rightarrow \|V_s\| \le \exp(-st\lambda - (1-s)t\mu)$$

$$\times \exp\left[\left(\frac{st\lambda^2}{a+\lambda}\right) + \left(\frac{(1-s)t\mu^2}{a+\mu}\right)\right]$$

$$= \exp\left[\frac{-st\lambda(a+\lambda) + st\lambda^2}{a+\lambda} + \frac{-(1-s)t\mu(a+\mu) + (1-s)t\mu^2}{a+\mu}\right]$$

$$= \exp\left[\frac{-ast\lambda}{a+\lambda} + \frac{-a(1-s)t\mu}{a+\mu}\right] \le 1$$

So $\|(e^{tA_\lambda} - e^{tA_\mu})x\| \le t\int_0^1 ds \,\|(A_\lambda - A_\mu)x\|, \; x \in X.$

If $x \in D(A)$, $A_\lambda x \to Ax \Rightarrow$

$$(e^{tA_\lambda} - e^{tA_\mu})x \to 0$$

as $\lambda, \mu \to \infty$. So $e^{tA_\lambda}x$ approaches a limit as $\lambda \to \infty$ for each $x \in D(A)$. Let $E_t x = \lim_{\lambda \to \infty} e^{tA_\lambda}x$. Now apply Lemma 10.2.

Suppose $a \leq 0$.

Let $B = A + a - 1$

$\lambda \in \rho(B) \iff \lambda - a + 1 \in \rho(A)$

So long as $\lambda - a + 1 \geq b$, $\lambda \in \rho(B)$

So there is a $b_1 > 0$ so that $[b_1, \infty) \subseteq \rho(B)$

and for $\lambda \geq b_1$, $\lambda - a + 1 \geq b$

$$\|(\lambda - B)^{-1}\| = \|(\lambda - a + 1 - A)^{-1}\|$$

$$\leq \frac{1}{a + (\lambda - a + 1)} = \frac{1}{1 + \lambda}.$$

So we get a family $\{E_t\}_{t \geq 0}$ of bounded operators

so that (a) $E_s E_t = E_{t,s}$ $\forall t \geq 0, s \geq 0$,

(b) $E(0) = I$, (d) $E_t x$ is continuous in t for

each $x \in X$, (f) $E_t(\lambda - A)^{-1} = (\lambda - A)^{-1} E_t$,

$\lambda \geq b_1, t \geq 0$,

$$\|E_t\| \leq e^{-t}$$

and $\dfrac{dE_t x}{dt} = BE_t x$, $x \in D(A)$, $t \geq 0$

Now let $F_t = e^{t-at} E_t$, $t \geq 0$.

Clearly (a), (b), (d) and (f) obtain for F_t.

Now $\|F_t\| = \|e^{t-at} E_t\| = e^{t-at} \|E_t\|$

$$\leq e^{t-at} e^{-t} = e^{-at}$$

and

$$\frac{dF_t x}{dt} = \frac{d}{dt}\left(e^{t-at} E_t x\right)$$

$$= (1-a) e^{t-at} E_t x + e^{t-at} \frac{d}{dt} E_t x$$

$$= (1-a) e^{t-at} E_t x + e^{t-at} B E_t x$$

$$= (1-a+B) e^{t-at} E_t x$$

$$= (1-a+B) F_t x$$

$$= (1-a+A+a-1) F_t x$$

$$= A F_t x$$